ON THE STABILITY OF NONLINEAR STOCHASTIC SYSTEMS

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Stability of systems with random parametric excitation was investigated by several workers. The cases when the stability towards white-noise excitation was investigated were the most successfull ones, since the methods of the theory of Markov processes (see eg. [1 to 4] e.a.) could be utilised. Investigation of stability under nonwhite excitation is much harder, and this is the reason why most authors limited their investigations to either linear [5 to 7] or nonlinear systems of some particular type [7]. In [8] we find the stability criteria for an arbitrary nonlinear system with excitations of any type, but this criterion is effective only in the cases when the solution is a Markov process.

Authors of [8 and 9] introduced the use of Liapunov function in problems of stability under random excitation. We shall however utilise that aspect of the Liapunov method, which was first used for similar purposes by the authors of [10 and 7].

1. We shall consider the system described by the following differential equation in vector form

$$dx / dt = G(x, t, \xi(t)) \tag{1.1}$$

Here x and G are vectors belonging to the l-dimensional Euclidean space E_l, while $\xi(t)$ is a random process which can assume values from the Euclidean space E_k.

We can assume without any loss of generality that $G(0, t, \xi(t)) \equiv 0$ and consider the problem of stability of the trivial solution $x(t) \equiv 0$. Following [9, 3 and 11], we shall introduce some definitions. We shall call the trivial solution of (1.1):

1°. Almost surely stable, if, for any $\varepsilon > 0$ and $\delta < 0$ such r can be found, that

$$P\{|x(t, x_0, t_0)| > \varepsilon\} < \delta \quad \text{when } t \geqslant t_0, \ |x_0| < r \tag{1.2}$$

2°. Almost surely asymptotically stable if it is almost surely stable and, if for any $\varepsilon > 0$ such $r = r(\varepsilon)$ can be found, that

$$P\{|x(t, x_0, t_0)| > \varepsilon\} \to 0 \quad \text{when } t \to \infty, \ |x_0| < r$$

3°. p-stable, if for any $\varepsilon > 0$ such $r > 0$ can be found, that

$$\langle |x(t, x_0, t_0)|^p \rangle < \varepsilon \quad \text{when } t \geqslant t_0, \ |x_0| < r \quad (p > 0)$$

Here and in the following the <> parentheses will denote a probabilistic mean (mathematical expectation).

4°. Asymptotically p-stable if it is p-stable and

$$\langle |x(t, x_0, t_0)|^p \rangle \to 0 \quad \text{as} \quad t \to \infty$$

5°. Almost surely stable in the large if it is almost surely stable and, if also for any x_0, ε, and δ, $T = T(x_0, \varepsilon, \delta)$ can be found such, that when $t > T$, then (1.2) is true. Asymptotic and p-stability in the large, are defined analogously.

6°. Exponentially p-stable if it is p-stable and if also constant $A > 0$ and $\alpha > 0$ exist such, that

$$\langle |x(t, x_0, t_0)|^p \rangle \leqslant A\, |x_0|^p \exp\{-\alpha\,(t - t_0)\}$$

7°. Stable with probability one in some sense or other, if all trajectories except perhaps a set of trajectories with probability zero, are stable in the corresponding sense.

Remark. In [7], another definition is used. There the system is asymptotically stable with probebility one, if for all initial conditions $x(t_0) = x_0$

$$P\{\lim_{t\to\infty} x(t, x_0, t_0) = 0\} = 1 \tag{1.3}$$

For the linear selfsimilar systems and steady processes $\xi(t)$, this definition is apparently equivalent to the definition found in [7]. It cannot however be used in the general case for two reasons. Firstly, examples exist of nonlinear determinate systems which satisfy the condition (1.3) but which, nevertheless, are not Liapunov stable in the classical sense, secondly, examples are easily found of linear systems, excited by a nonsteady random process, for which condition (1.3) is fulfilled, but for which, nevertheless, every trajectory is unstable with probability one.

Unless some initial restrictions are imposed on the system, we must not expect to obtain nontrivial and effective conditions of stability. In this paper we shall investigate conditions of stability of systems of the type

$$\frac{dx}{dt} = F(x, t) + \sigma(x, t)\,\xi(t), \qquad (F(0, t) \equiv 0;\quad \sigma(0, t) \equiv 0) \tag{1.4}$$

Here σ is a $k \times l$ matrix, x and $F \in E_l$, and $\xi(t) \in E_k$. Following the example of [7 and 10], we shall give the sufficient conditions of stability in terms of the existence of the Liapunov function of the reduced system

$$dx / dt = F(x, t) \tag{1.5}$$

With reference to all Liapunov functions $V(x, t)$ which will be encountered in this paper, we assume that they satisfy the Lifschitz condition in x

$$|V(x_2, t) - V(x_1, t)| < L\,|x_2 - x_1| \tag{1.6}$$

in every bounded region. If L is independent of the region, i.e.

$$\sup_{x,\,t} \frac{|V(x_2, t) - V(x_1, t)|}{|x_2 - x_1|} = L < \infty$$

then we shall use the notation $V \in C(L)$. We shall also adopt the notation

$$\|\sigma\| = \left(\sum_{i=1}^{k} \sum_{j=1}^{l} \sigma_{ij}^2\right)^{1/2}$$

2. As we know, the process $\xi(t)$ satisfies the law of large numbers, if for any

$\varepsilon > 0$ and $\delta > 0$, $T > 0$ can be found such, that

$$P\left\{\left|\frac{1}{t}\int_0^t \xi(s)\,ds - \frac{1}{t}\int_0^t \langle \xi(s) \rangle\,ds\right| > \delta\right\} < \varepsilon \quad \text{when} \quad t > T$$

If, on the other hand,

$$P\left\{\frac{1}{t}\int_0^t \xi(s)\,ds - \frac{1}{t}\int_0^t \langle \xi(s) \rangle\,ds \to 0 \quad \text{when} \quad t \to \infty\right\} = 1$$

then the process $\xi(t)$ satisfies a more rigorous law of large numbers.

Sufficiently wide conditions of applicability of the law of large numbers to random processes, are given in [12]. The fact that more rigorous law of large numbers can be utilised to establish the stability of a system with probability one was first mentioned in [5]. Development of this idea is found in [7]. We shall show, that the law of large numbers in its weak form leads, under additional conditions, to the almost sure stability.

Theorem 2.1. Let us assume, that a Liapunov function $V \in C(L)$, exists for the system (1.5), satisfying the conditions ($c_i > 0$ are constant)

$$\inf_{t>0,\ |x|>r} V(x,t) = V_r > 0 \quad \text{when} \quad r > 0 \tag{2.1}$$

$$\frac{d^\circ V}{dt} \leqslant -c_1 V, \qquad \|\sigma(x,t)\| \leqslant c_2 V \tag{2.2}$$

Here and in the following

$$\frac{d^\circ V(x,t)}{dt} = \overline{\lim_{h\to +0}}\ \frac{1}{h}\left[V(x(t+h,x,t),t+h) - V(x,t)\right]$$

is the derivative of V by virtue of the system (1.5).

Then, the trivial solution of the system (1.4) is almost surely asymptotically stable in the large for any process $\xi(t)$, for which

$$\sup_{t>0} \langle |\xi(t)| \rangle < \frac{c_1}{Lc_2} \tag{2.3}$$

and provided that the process $|\xi(t)|$ satisfies the law of large numbers. If, on the other hand, the process $|\xi(t)|$ satisfies the sharper law of large numbers, then the same conditions secure the asymptotic stability in the large of the solution $x = 0$, with probability one.

Proof. Let $x^0(t)$ be the solution of Equation (1.4), satisfying the initial condition $x^\circ(0) = x_0$. Then, accounting for the conditions of the theorem, we easily obtain

$$\frac{dV(x^\circ(t),t)}{dt} \leqslant \frac{d^\circ V(x^\circ(t),t)}{dt} + Lc_2|\xi(t)|V \leqslant V(-c_1 + Lc_2|\xi(t)|)$$

from which the inequality

$$V(x^\circ(t),t) \leqslant V(x_0,0)\exp\left\{Lc_2 t\left(\frac{1}{t}\int_0^t |\xi(s)|\,ds - \frac{c_1}{Lc_2}\right)\right\} \tag{2.4}$$

follows. Let now make $\varepsilon > 0$ and $\delta > 0$ arbitrary. Using (2.3) together with the fact that the process $\xi(t)$ satisfies the law of large numbers, we shall choose $T > 0$ such, that when $t \geqslant T$, then the inequality

$$P\left\{\frac{1}{t}\int\limits_0^t |\xi(s)|\, ds > \frac{c_1}{Lc_2}\right\} < \varepsilon \tag{2.5}$$

is fulfilled. Now, let us choose a number $M > 1$ large enough to ensure that

$$P\left\{Lc_2 \int\limits_0^T |\xi(s)|\, ds > \ln M\right\} < \varepsilon \tag{2.6}$$

Finally, let us choose r small enough to ensure that

$$V(x_0, 0)\, M < V_\delta \quad \text{for } |x_0| < r \tag{2.7}$$

From (2.4) to (2.7) we obtain, considering the cases $t < T$ and $t \geqslant T$ separately, that for $|x_0| < r$ and all $t > 0$,

$$P\{|x(t)| > \delta\} \leqslant P\{V(x(t), t) > V_\delta\} \leqslant \varepsilon$$

This, together with

$$P\left\{\frac{1}{t}\int\limits_0^t |\xi(s)|\, ds > \frac{c_1}{Lc_2}\right\} \to 0 \quad \text{при } t \to \infty$$

proves the first assertion of the theorem. Second part is proved analogously.

Theorem 2.2. Let a Liapunov function $V(x, t) \in C(L)$, exist, for which the relations (3.1) and (2.2) hold together with the inequality (for some $c > 0$)

$$V(x, t) > c\,|x| \tag{2.8}$$

for the system (1.4), and let the process $\xi(t)$ be such, that for some positive constants k_1 and k_2 and all $t > t_0$

$$\left\langle \exp\left\{k_1 \int\limits_{t_0}^t |\xi(s)|\, ds\right\}\right\rangle \leqslant \exp\{k_2(t - t_0)\} \tag{2.9}$$

where the constants k_i, c_i and L, are connected by

$$Lk_2c_2 \leqslant k_1c_1 \tag{2.10}$$

Then, the solution $x(t) \equiv 0$ of the system (1.4) is p-stable, when $p \leqslant k_1/(Lc_2)$.

If, on the other hand, a stronger inequality

$$Lk_2c_2 < k_1c_1 \tag{2.11}$$

holds, then for the same p, the solution is exponentially p-stable. Proof of this theorem is based on the previously obtained inequality (2.4). Raising its both parts to the power k_1/Lc_2 and evaluating the mathematical expectations, we obtain, taking (2.8) into account,

$$\begin{aligned} c^{k_1/Lc_2}\langle |x(t)|^{k_1/Lc_2}\rangle &\leqslant \langle V(x, t), t)]^{k_1/Lc_2}\rangle \leqslant \\ &\leqslant [V(x_0, t_0)]^{k_1/Lc_2}\left\langle \exp\left\{k_1\int\limits_{t_0}^t |\xi(s)|\, ds - \frac{c_1k_1}{Lc_2}(t - t_0)\right\}\right\rangle \end{aligned} \tag{2.12}$$

which, together with the relations (2.9) to (2.11), yields the proof of our theorem.

3. Since in real systems random perturbations result from a large number of factors, each of them exerting little influence, it is natural to assume that the process $\xi(t)$ in (1.4) is Gaussian. We know [12], that such a process is uniquely characterised by its vector

of mathematical expectation $m(t) = \langle \xi(t) \rangle$ and a convariant matrix

$$K(s, t) = \operatorname{cov}(\xi(s), \xi(t)) = ((\langle [\xi_i(s) - m_i(s)]\, [\xi_j(t) - m_j(t)] \rangle))$$

Stability of linear systems under Gaussian parameter variation was investigated in [13 and 6]. In [6] the analysis was based on the estimate of the type (2.9) established for a Gaussian steady process satisfying the condition of 'mixing' sufficiently strongly. It was found, that the conditions under which the above estimate was valid, could be relaxed and simplified. We can say more precisely, that, when the conditions

$$|\langle \xi(t) \rangle| \leqslant a_0, \qquad \langle |\xi(t) - m(t)|^2 \rangle \leqslant a_1, \qquad \int_{t_0}^{t_1} \|K(s, u)\| ds \leqslant a_2 \tag{3.1}$$

are fulfilled for the Gaussian process for any t, t_0 and t_1, then the inequality

$$\left\langle \exp\left\{k_1 \int_{t_0}^{t_1} |\xi(s)|\, ds\right\}\right\rangle \leqslant \exp\left\{k_1\left(a_0 + \sqrt{a_1} + \frac{k_1 a_2}{2}\right)(t_1 - t_0)\right\} \tag{3.2}$$

is valid. This, together with the theorem 2.2 infers, that the system (1.4) is exponentially p-stable for sufficiently small p, provided it has a Liapunov function $V(x, t) \in C(L)$, satisfying (2.1), (2.2) and (2.8), and that $\xi(t)$ is a Gaussian process for which, for sufficiently small a_0 and a_1, conditions (3.1) hold.

Let us now apply the theorems proved in section 2, to linear systems of the type

$$\frac{dx}{dt} = (A(t) + \eta(t))\, x \tag{3.3}$$

where the elements of a square matrix $\eta(t)$ are random processes, and the system under investigation

$$\frac{dx}{dt} = A(t)\, x \tag{3.4}$$

is exponentially stable.

By the Malkin theorem ([14], p. 313), the exponential stability of (3.4) implies that a positive definite quadratic form $(C(t), x, x) = W(t, x)$, exists for the system, for which

$$\frac{d^\circ W}{dt} \leqslant -\lambda |x|^2 \qquad (\lambda > 0)$$

System (3.3) must be written in the form (1.4), before the theorem 2.1 can be applied to it. This is easily done by substituting an $l^2 \times l$ matrix

$$\sigma(x, t) = \begin{Vmatrix} x_1 & x_2 & \cdots & x_l & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_1 & 0 & \cdots & x_l & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & x_1 & x_2 & \cdots & x_l \end{Vmatrix}$$

for $\sigma(x, t)$, and

$$\eta_{ik}(t) = \xi_{(i-1)l+k}(t) \quad \text{where} \quad (\xi(t) \quad \text{is a vector from } E_{l^2}$$

Considering the Liapunov function $V(x, t) = (W(x, t))^{1/2}$ and applying Theorem 2.1, we obtain.

Theorem 3.1. Let the system (3.4) be exponentially stable. Then, the system (3.3) is almost surely asymptotically stable for any (matrix) random process $\eta(t)$ such, that the process $\|\eta(t)\|$ satisfies the law of large numbers and $\langle\|\eta(t)\|\rangle < c$, where c is a sufficiently small constant. If, on the other hand, the process $\|\eta(t)\|$ satisfies the sharper law of large numbers, then the same conditions secure the asymptotic stability of (3.3) in the large with probability one.

Next we shall consider the conditions of p-stability of the linear system (3.3), and we shall limit ourselves to the case where A is a constant stability matrix (i.e. Re $\lambda_i < 0$), λ_i are the eigenvalues of A and $\eta(t)$ is a Gaussian random process. We shall first consider the case when the equation is given in E_1.

Example. Let the equation

$$\frac{dx}{dt} = (a + \eta(t))\,x, \qquad x(0) = x_0 \tag{3.5}$$

be given in E_1. Here $\eta(t)$ is a Gaussian steady process with $\langle\eta(t)\rangle = 0$ and $K(t-s) = \langle\eta(s)\eta(t)\rangle$ is a correlation function. The fact that the integral of a Gaussian process also possesses a Gaussian probability distribution, infers that

$$\langle |x(t)|^p\rangle = |x_0|^p \left\langle \exp\left\{apt + p\int_0^t \eta(s)\,ds\right\}\right\rangle = |x_0|^p \exp\left\{apt + \frac{p^2}{2}\int_0^t\int_0^t K(u-s)\,du\,ds\right\} \tag{3.6}$$

If the function $K(u)$ is integrable absolutely, then we know [12] that the process ξ kt) has a bounded spectral density $f(\lambda)$, and

$$\int_0^t\int_0^t K(u-s)\,du\,ds = f(0)\,t + o(t) \tag{3.7}$$

from which, together with (3.6) it follows, that the solution $x(t) \equiv 0$ of (3.5) is asymptotically p-stable when $p < -2a\,/\,f(0)$, provided $a < 0$. When $a > 0$, the insability of the solution follows from its explicit form. If $a = 0$, then the solution $x(t) \equiv 0$ will be unstable if, for example, $f(0) \neq 0$. Indeed in this case (3.7) implies that

$$D(t) = \int_0^t\int_0^t K(u-s)\,du\,ds = \left\langle\left[\int_0^t \xi(s)\,ds\right]^2\right\rangle \to \infty \quad \text{as} \quad t \to \infty \tag{3.8}$$

and

$$P\left\{\int_0^t \xi(s)\,ds > \sqrt{D(t)}\right\} = 1 - \Phi(1) \tag{3.9}$$

Here $\Phi(x)$ is a distribution function of a normal law, with parameters 0 and 1. From (3.8) and (3.9), the instability of the trivial solution in this case, follows.

Next we shall consider the case of $x \in E_l$. If A is a stability matrix, then [14] there exist a symmetric, positive definite matrix C, such that the matrix $CA + A^*C$ is negative definite. Let us denote by λ the largest positive number for which,

$$\frac{d^\circ(Cx, x)}{dt} = ((CA + A^*C)\,x, x) \leqslant -\lambda\,(Cx, x) \tag{3.10}$$

for all $x \in E_l$. The following estimate for λ is easily obtained:

$\lambda > -\lambda_{\max}^{d} / \lambda_{\max}^{c} > 0$, where $\lambda_{\max}^{c}$ and $\lambda_{\max}^{d}$ are the largest eigenvalues of the matrices C and $D = CA + A^{*}C$ respectively.

Theorem 3.2. Let A be an $l \times l$ stability matrix, C a positive definite matrix, satisfying the condition (3.8) and let $\eta(t) = ((\eta_{ij}(t)),\ i, j = 1, \ldots, l$ be a Gaussian random process. We assume that for the process $\eta^{\circ}(t) = C^{1/2}\eta(t)\, C^{-1/2}$, conditions

$$|\langle\eta^{\circ}(t)\rangle| \leqslant a_0; \qquad \langle|\eta^{\circ}(t) - \langle\eta^{\circ}(t)\rangle|^2\rangle \leqslant a_1, \qquad \int_{t_0}^{t_1} \|K(s, u)\|\, ds \leqslant a_2$$

where $K(s, t) = \operatorname{cov}(\eta^{\circ}(s), \eta^{\circ}(t))$ is an $l^2 \times l^2$ matrix, are satisfied.

Then, the trivial solution of (3.3) is asymptotically p-stable for $p < [\lambda - 2(a_0 + \sqrt{a_1})] / 2$, if $\lambda > 2(a_0 + \sqrt{a_1})$.

Proof. By (3.8), we have for the Liapunov function $V(x) = (Cx, x)$

$$\frac{dV(x(t))}{dt} \leqslant -\lambda V + ((C\eta + \eta^{*}C)x, x) \leqslant V(-\lambda + 2\|\eta^{\circ}(t)\|) \tag{3.11}$$

Here we have used the estimate

$$(C\eta x, x) = (C^{1/2}\eta c^{-1/2}C^{1/2}x,\ C^{1/2}x) \leqslant \|C^{1/2}\eta C^{-1/2}\|\|C^{1/2}x\|^2 \leqslant \|\eta^{\circ}\|(Cx, x)$$

hence

$$[V(x(t))]^p \leqslant [V(x_0)]^p \exp\left\{-p\lambda t + 2p\int_0^t \|\eta^{\circ}(s)\|\, ds\right\}$$

From this, computing the mathematical expectation and using the estimate (3.2), we obtain

$$\langle[V(x(t))]^p\rangle \leqslant [V(x_0)]^p \exp\{pt(-\lambda + 2a_0 + 2\sqrt{a_1} + pa_2)\} \tag{3.12}$$

from which the proof follows directly.

Note 3.1. We can see from the example that in the one-dimensional case, fulfilment of the condition (3.10) secures the stability of the system for any $a_1 = \sup\langle|\eta(t)|^2\rangle$, provided $a_0 = 0$. It can easily be illustrated, that in the multidimensional case the above statement is usually not true; noise of sufficient intensity can nullify the stability.

3.2. Our example shows also, that an unstable one-dimensional system remains unstable under the action of Gaussian noise with the zero mean. It can be shown, that this property is also not transferable to the multidimensional case, i.e. multidimensional unstable systems can be stabilised by means of Gaussian noise. (The last statement can be found in [13], but, as noted in [4], its proof is incorrect).

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